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A system of parallel conductors in an external field

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Abstract. We present and discuss the general solution of the electrostatic problem of a system of parallel cylindrical conductors in the presence of an external field and in particular in a homogeneous external field. For this purpose we make extensive use of the theory of automorphic functions. We determine the polarisation of the system.

1. Introduction

There has recently been an increasing interest in the solution of the electromagnetic problem for a system of parallel cylindrical conductors. Let us mention as examples the calculation of proximity effects in multiwire cables (Belevitch 1977) and the determination of propagation parameters in uniform cables (Lenahan 1977) (see also Paul and Feather 1976). The electromagnetic problem of parallel cylindrical conductors can be solved in a very general way with the help of the theory of automorphic functions (Alessandrini *et al* 1974). The desired solution is found by means of a generalisation of the method of images; these are generated in a systematic way using the group of automorphisms associated with the given problem. Related electromagnetic problems, such as the skin effect in multiconductor systems (Alessandrini *et al* 1976), can be treated in a similar way.

Our main purpose here is to present and discuss the general solution of the electrostatic problem of a system of parallel conductors in an external dipole field (Burnside 1891). The particular case of a homogeneous external field may be relevant for the design of multiwire proportional chambers. The dipole moment of the system is relevant in the treatment of artificial dielectrics, either ordered (Collin 1960) or random (Levine and McQuarrie 1968).

In § 2 we briefly review the necessary elements of the theory of automorphic functions connected with our problem. The cases of one and two conductors in an external field are treated explicitly in § 3 so as to show the potentialities of our approach. These results are then used to improve the convergence of the series in the general case. Finally, § 4 is devoted to the computation and discussion of the dipole moment of the system and to state our conclusions.

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2. The group of automorphisms

We begin our discussion by introducing a generalisation of the method of images. As is well known, the image of a point z in the complex plane C, with respect to any circle, is not an analytic function of z. However, a pair of inversions with respect to two circles is indeed an analytic function of z. This double inversion has the form

$$z' = T(z) = (az+b)/(cz+d), \tag{2.1}$$

which is called a projective transformation (Forsythe 1918). The coefficients a, b, c, d are complex numbers on which we impose the restriction

$$ad - bc = 1. \tag{2.2}$$

This equation, together with the geometry of the problem (position and radii of the circles), completely determines the values of the four constants. The set of all projective transformations is a group isomorphic to SL(2, C).

Two important properties of projective transformations are that they map circles onto circles and that they preserve anharmonic ratios.

We define the fixed points of a transformation by the equation

$$T(z) = z. \tag{2.3}$$

The projective transformation (2.1) has two fixed points ξ and η.

If $\xi \neq \eta$, then the transformation

$$z' = t(z) = \frac{z/(\xi-\eta)^{1/2} - \xi/(\xi-\eta)^{1/2}}{z/(\xi-\eta)^{1/2} - \eta/(\xi-\eta)^{1/2}} \tag{2.4}$$

maps (ξ, η) onto $(0, \infty)$, and (2.1) acts on z' as a scale transformation

$$T'(z') = tTt^{-1}(z') = Kz', \tag{2.5}$$

where K is the multiplier of T. In the following, we shall choose (ξ, η) in such a way that $|K| > 1$.

Now we come to the problem of $N+1$ parallel cylindrical conductors. They define, in the complex plane C, a set of $N+1$ non-intersecting circles. We begin by forming the inverses of the circles $1, 2, \ldots, N$ with respect to the one labelled 0, and then construct N transformations T_i mapping each circle onto its inverse. A group of projective transformations, called the group of automorphisms of the domain under consideration, can be built with the T_i and their inverses T_i^{-1}; all binary products T_1^2, T_1T_2, $T_1T_3, \ldots, T_1T_2^{-1}, \ldots$; all ternary products such as T_i^3, $T_1T_2^2, \ldots$; and so on. (For details see Alessandrini *et al* (1974).) We shall call T_i the generators of the group of automorphisms and shall label with T_α any element of the group.

As T_i is the product of inversions with respect to the circles i and 0, the set of transformations $\{T_\alpha\}$ applied to a given point z generates only half of the images. The other half is generated by application of $\{T_\alpha\}$ to $\bar{z}$, the inverse of z with respect of the circle 0.

The group of automorphisms provides the formal definition of the generalised method of images. Using it, we shall find series for the Green function involving its elements T. We shall be faced with the so-called automorphic functions, defined by the property

$$\phi[T_\alpha(z)] = \phi(z). \tag{2.6}$$

The theory of these automorphic functions (developed in Burnside 1891) is based on the Poincaré θ series

$$\theta(z, s) = \sum_\alpha (c_\alpha z + d_\alpha)^{-2}(T_\alpha(z) - s)^{-1}, \tag{2.7}$$

where $(c_\alpha z + d_\alpha)^{-2}$ is a convergence factor. This is actually an automorphic form of dimension two, having the property

$$\theta[T_\beta(z), s] = (c_\beta z + d_\beta)^2 \theta(z, s), \tag{2.8}$$

but automorphic functions can be built out of it. It has been proved (Burnside 1891) that there exist only N independent θ functions for a domain of multiplicity N.

The main tool for the solution of electrostatic problems related to the system of circles is the Green function of the domain, which can be constructed immediately from the generalised method of images. For isolated conductors it is given by

$$\omega_{s,\bar{s}}(z) = \sum_\alpha \ln\left(\frac{(T_\alpha(z) - s)(T_\alpha(z_0) - \bar{s})}{(T_\alpha(z) - \bar{s})(T_\alpha(z_0) - s)}\right), \tag{2.9}$$

where s is the position of the source point, $\bar{s}$ its inverse with respect to the circle 0, and z_0 the normalisation point where $\omega_{s,\bar{s}}(z_0) = 0$. Expression (2.9) coincides with the definition of the third Abelian integral of the domain in the complex normalisation.

We shall be mainly interested in the potential and the field distribution generated by a dipole line, and in particular those generated by a uniform field. This potential can be found from the Green function in the following way:

$$\begin{aligned}\Phi_{s,\bar{s}}(z) &= \boldsymbol{p} \cdot \nabla \omega_{s,\bar{s}}(z) = \tfrac{1}{2}(p\, \partial/\partial s + p^* \,\partial/\partial \bar{s})\omega_{s,\bar{s}}(z) \\ &= -p \sum_\alpha \left(\frac{1}{T_\alpha(z) - s} - \frac{1}{T_\alpha(z_0) - s}\right) \\ &\quad - \frac{R^2}{(S^* - J^*)^2} p^* \sum_\alpha \left(\frac{1}{T_\alpha(z) - \bar{s}} - \frac{1}{T_\alpha(z_0) - \bar{s}}\right),\end{aligned} \tag{2.10}$$

where p is the external dipole moment per unit length, and R and J are the radius and centre of the circle 0.

The function

$$\Psi_s(z) = \sum_\alpha \left(\frac{1}{T_\alpha(z) - s} - \frac{1}{T_\alpha(z_0) - s}\right) \tag{2.11}$$

is called the second Abelian integral of the domain (Burnside 1891). The complex potential of a dipole can be written in terms of second Abelian integrals as

$$\Phi_{s,\bar{s}}(z) = -p\Psi_s(z) - [R^2/(S^* - J^*)]p^*\Psi_{\bar{s}}(z). \tag{2.12}$$

The particular case of the uniform external field is obtained in the limit $s \to \infty$, $p \to \infty$ and $p = E^* s^2$. In this way we obtain

$$\Phi_E(z) = E^* \sum_\alpha (T_\alpha(z) - T_\alpha(z_0)) - ER^2 \sum_\alpha \left(\frac{1}{T_\alpha(z) - J} - \frac{1}{T_\alpha(z_0) - J}\right), \tag{2.13}$$

where E stands for the uniform external field.

Both (2.10) and (2.13) are rapidly convergent series which satisfy the boundary conditions. The potentials are given by the sum of the potentials generated by all the images of the dipole; the functions involved have only simple poles (singularities characteristic of dipole lines in two dimensions) inside the circles.

Let us finally remark that we can build up other Green functions with different boundary conditions.

For instance, a Green function for grounded conductors can be found by adding to equation (2.9) a suitable set of first Abelian integrals. These latter functions represent the potentials due to charged conductors (Alessandrini *et al* 1974).

3. Discussion of simple cases

Let us first consider the case of a simple circle in an external field. The group of automorphisms reduces to the identity and we immediately obtain

$$\Phi_E(z) = E^*(z - z_0) - ER^2[1/(z - J) - 1/(z_0 - J)], \tag{3.1}$$

which is the well-known elementary solution.

As a second example, we consider the case of two identical cylinders in the field of a dipole line at the point s. With the geometry shown in figure (1), the only generator of the group of automorphisms is

$$T = \begin{pmatrix} 2J^2/R^2 - 1 & 2J(1 - J^2/R^2) \\ -2J/R^2 - 1 & 2J^2/R^2 - 1 \end{pmatrix}, \tag{3.2}$$

whose fixed points are

$$\xi = +(J^2 - R^2)^{1/2}, \qquad \eta = -(J^2 - R^2)^{1/2}, \tag{3.3}$$

and whose multiplier is

$$K = [(J + \xi)/(J - \xi)]^2. \tag{3.4}$$

In this simple configuration the set of images is generated by repeated reflection on both circles.

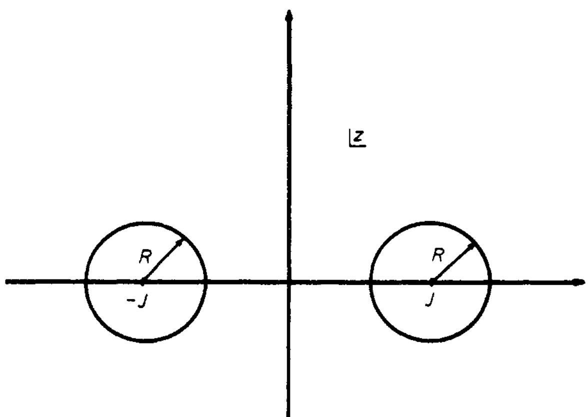

Figure 1. Two identical cylinders in the field of a dipole line at the point s.

We can now write the solution of this problem using equation (2.10):

$$\Phi_{s,\bar{s}}(z)=-p\sum_{n=-\infty}^{\infty}\left(\frac{1}{T^n(z)-s}-\frac{1}{T^n(z_0)-s}\right)$$
$$-\frac{R^2}{(S^*-J^*)^2}p^*\sum_{n=-\infty}^{\infty}\left(\frac{1}{T^n(z)-\bar{s}}-\frac{1}{T^n(z_0)-\bar{s}}\right). \tag{3.5}$$

We shall next find a compact expression for this potential in terms of Jacobi theta functions (Whittaker and Watson 1927). Let us perform the transformation (2.4) on our solution (3.5), taking into account that, under a projective transformation

$$z'=(Az+B)/(Cz+D), \tag{3.6}$$

the second Abelian integral transforms as

$$\Psi_s(z)=(Cs+D)^2\Psi'_{s'}(z'), \tag{3.7}$$

due to the properties of the Poincaré θ series. For our case,

$$\Psi'_{s'}(z')=\sum_{n=-\infty}^{\infty}\left(\frac{1}{K^n z'-s'}-\frac{1}{K^n z'_0-s'}\right). \tag{3.8}$$

Define now the variable u by

$$z'=s'\exp(-2\mathrm{i}u), \tag{3.9}$$

and the second Abelian integral (3.8) becomes

$$\Psi'_{s'}(u)=\frac{1}{s'[\exp(-2\mathrm{i}u)-1]}+\frac{2\mathrm{i}}{s'}\sum_{n=1}^{\infty}\frac{\sin(2u)K^{-n}}{1-2\cos(2u)K^{-n}-K^{-2n}}$$
$$-\left(\frac{1}{s'[\exp(-2\mathrm{i}u_0)-1]}+\frac{2\mathrm{i}}{s'}\sum_{n=1}^{\infty}\frac{\sin(2u_0)}{1-2\cos(2u_0)K^{-n}-K^{-2n}}\right). \tag{3.10}$$

This series can be summed using the expansion for the logarithmic derivative of the Jacobi theta function (Whittaker and Watson 1927):

$$\Psi'_{s'}(u)=\frac{\mathrm{i}}{2s'}\left(\frac{\theta'_1(u)}{\theta_1(u)}-\frac{\theta'_1(u_0)}{\theta_1(u_0)}\right). \tag{3.11}$$

With this result, our potential can be cast into the form

$$\Phi_{s,\bar{s}}(u)=-\frac{(1-s')^2}{\xi-\eta}p\Psi'_{s'}(u)-\frac{(1-\bar{s}')^2}{\xi-\eta}\frac{R^2p^*}{(S^*-J^*)^2}\Psi'_{s'}(u) \tag{3.12}$$

Finally, taking the limit $s\to\infty$ ($s'\to 1$), the complex potential of two identical cylinders in an external uniform field becomes

$$\Phi_E(u)=E^*(\eta-\xi)\frac{\mathrm{i}}{2}\left(\frac{\theta'_1(u)}{\theta_1(u)}-\frac{\theta'_1(u_0)}{\theta_1(u_0)}\right)+E(\eta-\xi)\frac{\mathrm{i}}{2}\left(\frac{\theta'_1(u-\pi\tau/2)}{\theta_1(u-\pi\tau/2)}-\frac{\theta'_1(u_0-\pi\tau/2)}{\theta_1(u_0-\pi\tau/2)}\right), \tag{3.13}$$

where τ (quarter period of the Jacobi theta functions) is defined by

$$K=\exp(-2\pi\mathrm{i}\tau). \tag{3.14}$$

In order to show that this result agrees with the already known solution (Morse and Feshbach 1961), let us introduce the variable

$$\omega = \mathrm{i}\pi + 2\mathrm{i}u. \tag{3.15}$$

If we substitute (3.15) in equation (3.13) and use the Fourier series of logarithmic derivatives of the theta functions (Whittaker and Watson 1927), we find

$$\Phi_E(\omega) = \mathrm{i}E\left[1 + 2\sum_{n=1}^{\infty}(-)^n\left(\frac{\sinh(n\omega)}{\sinh(2n\xi_0)} - \frac{\sinh n(\omega - 2\xi_0)}{\sinh(2n\xi_0)}\right)\right], \tag{3.16}$$

where ξ_0 is the dipolar coordinate of the surface of the cylinder. The real part of this analytic function coincides with the solution given by Morse and Feshbach (1961).

We end this section with a remark concerning the general case of $N+1$ conductors. Obviously, the summation method used in the case of two conductors can be applied in the general case to any group element T_α. The calculations lead to a compact expression for the second Abelian integral:

$$\Psi'_{s'}(z') = \frac{1}{z'-s'} - \frac{1}{z'_0 - s'} + \frac{\mathrm{i}}{2}\sum_{\beta\in M}\frac{(1-S_\beta)^2}{\xi_\beta - \eta_\beta}$$
$$\times\left(\frac{\theta'_1(u_\beta)}{\theta_1(u_\beta)} - \frac{\theta'_1(u_{0\beta})}{\theta_1(u_{0\beta})} - \cot u_\beta + \cot u_{0\beta}\right), \tag{3.17}$$

where the sum extends over the set M of elements of the group which are not powers of any other element. This last form of the series represents a great improvement in the convergence whenever an element of the group becomes 'near parabolic' (Alessandrini *et al* 1974).

4. Dipole moment

We shall now compute the dipole moment per unit length of our system of parallel conductors (or discs). Let us first recall that the dipole moment per unit length of a system of line charges is defined as the coefficient of the term $1/z$ in the asymptotic expansion of the potential, and the susceptibility of the system is the derivative of the dipole per unit length with respect to the external field.

The total dipole moment of our system of circles will be given by

$$P = (1/2\pi\mathrm{i})\oint \Phi_E(z)\,\mathrm{d}z, \tag{4.1}$$

where the integration path is a contour encircling all the conductors.

As the only singularities of the potential are simple poles inside the conductors, we can immediately write an alternative form for the dipole moment as a sum of the dipole moments of all the images:

$$P = \sum_\alpha \operatorname{res}(\Phi_E(z_\alpha)), \tag{4.2}$$

where the z_α are the locations of all the images of the dipole at infinity,

$$z_\alpha = \begin{cases} T_\alpha(\infty) \\ T_\alpha(\bar{\infty}) \end{cases}. \tag{4.3}$$

For the general case of $N+1$ conductors in homogeneous external field E, we have from equation (3.17) the expression

$$\Phi_E(z)=E^*(z-z_0)-\frac{\mathrm{i}E^*}{2}\sum_{\beta\in M}(\xi_\beta-\eta_\beta)\left(\frac{\theta_1'(u_\beta)}{\theta_1(u_\beta)}-\cot(u_\beta)-\frac{\theta_1'(u_{0\beta})}{\theta_1(u_{0\beta})}+\cot(u_{0\beta})\right)$$
$$-\frac{E^*R^2}{z-J}-\frac{\mathrm{i}ER^2}{2}\sum_{\beta\in M}\frac{\xi_\beta-\eta_\beta}{(J-\eta_\beta)(J-\xi_\beta)}$$
$$\times\left(\frac{\theta_1'(\tilde{u}_\beta)}{\theta_1(\tilde{u}_\beta)}-\cot(\tilde{u}_\beta)-\frac{\theta_1'(\tilde{u}_{0\beta})}{\theta_1(\tilde{u}_{0\beta})}+\cot(\tilde{u}_{0\beta})\right). \tag{4.4}$$

Equation (4.4) reduces to (3.13) for the case of a pair of cylinders.

The poles of the potential are the zeros of the Jacobi theta functions, located at the points

$$\hat{u}=n\pi+m\pi\tau, \tag{4.5}$$

with n and m integers. The computation of the residues is very simple, and we find that

$$\sum \operatorname{res}(\theta_1'(u)/\theta_1(u)-\cot u)$$
$$=\frac{2}{\mathrm{i}}(\xi-\eta)\sum_n q^{2n}\left(\frac{C}{(1-Cq^{2n})^2}-\frac{1/C}{(1-q^{2n}/C)^2}\right), \tag{4.6}$$

where C is defined by

$$C=(J-\eta_\beta)/(J-\xi_\beta), \tag{4.7}$$

so that

$$\exp(-2\mathrm{i}\tilde{u}_\beta)=(z-\xi_\beta)C/(z-\eta_\beta), \tag{4.8}$$

and the corresponding relation for u is found by setting $C=1$ in (4.8). Introducing now

$$C=\exp(2\mathrm{i}r), \tag{4.9}$$

one finds that

$$(\mathrm{d}/\mathrm{d}r)(\theta_1'(r)/\theta_1(r)-\cot(r))=\sum \operatorname{res}(\theta_1'(u)/\theta_1(u)-\cot(u)), \tag{4.10}$$

and using well-known properties of the Jacobi theta functions one finally obtains

$$P=-ER^2-\frac{E^*}{4}\sum_{\beta\in M}\frac{(\xi_\beta-\eta_\beta)^2}{3}\left(\frac{\theta_1'''(0)}{\theta_1'(0)}+1\right)$$
$$-\frac{ER^2}{4}\sum_{\beta\in M}\frac{(\xi_\beta-\eta_\beta)^2}{(J-\eta_\beta)(J-\xi_\beta)}\left(\frac{\theta_1''(r)}{\theta_1(r)}-\frac{\theta'^2(r)}{\theta_1^2(r)}+\frac{1}{\sin^2(r)}\right). \tag{4.11}$$

For the particular case of a pair of cylinders we have $r=-\pi\tau/2$, and only one term survives in the series (4.11). The result is

$$P_2=-E\left\{R^2+\xi^2\left[\frac{\theta_4''(0)}{\theta_4(0)}-\left(\frac{K^{1/4}-K^{-1/4}}{2}\right)^{-2}\right]\right\}-E^*\frac{\xi^2}{6}\left(\frac{\theta_1'''(0)}{\theta_1'(0)}+1\right). \tag{4.12}$$

The susceptibility tensor can be found by taking the derivative with respect to the external field in this last expression.

Let us finish with some remarks concerning our results. The series of Burnside for the Abelian integrals of the given multiply connected domain, related to the system of cylinders in an external field, has here been shown to be a consequence of a generalisation of the method of images. The group properties embedded in this method were used to find a partial summation of the series, and to express the sums in terms of Jacobi theta functions. This result represents a great improvement in the convergence properties. The dipole moment of the system was found from the above-mentioned solution and could also be expressed as a series of Jacobi theta functions. The polarisation of the system of conducting cylinders can be found from these results. These are the main ingredients for the study of artificial dielectrics, a problem which will be taken up in forthcoming papers.

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